Discussion 19 Worksheet

Surface integrals of vector fields

Date: 11/17/2021

MATH 53 Multivariable Calculus

1 Vector Surface Integrals

Compute the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}$$

for a given vector field $\vec{F}(x, y, z)$ over the surface S.

(a) $\vec{F}(x, y, z) = \langle xy, yz, zx\rangle$ over the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$.

(b) $\vec{F}(x, y, z) = \langle x, y, z^2\rangle$ and S is the unit sphere centered at the origin.

2 Oriented Surfaces

Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$ for the given vector field $\vec{F}$ and oriented surface S. For closed surfaces, use the positive (outward) orientation.

(a) $\vec{F}(x, y, z) = \langle ze^{xy}, -3ze^{xy}, xy\rangle$. S is the parallelogram $x = u+v, y = u-v, z = 1+2u+v, 0 \le u \le 2, 0 \le v \le 1$ oriented upwards.

(b) $\vec{F}(x, y, z) = \langle 0, y, -z\rangle$ and S consists of the paraboloid $y = x^2 + z^2, 0 \le y \le 1$, and the disk $x^2 + z^2 \le 1, y = 1$.

(c) Do (b) using the divergence theorem.

(d) $\vec{F}(x, y, z) = \langle x^2, y^2, z^2\rangle$ and S is the boundary of the solid half cylinder $0 \le z \le \sqrt{1 - y^2}, 0 \le x \le 2$.

(e) Do (d) using the divergence theorem.

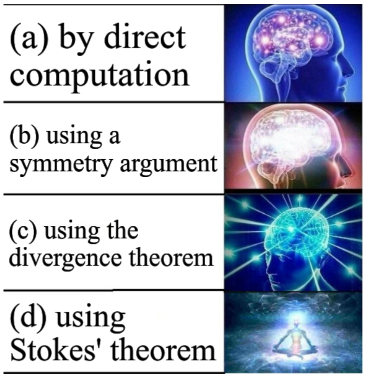
3 Divergence theorem?

- Let S be the same cylinder $x^2 + y^2 = 1, -1 \le z \le 1$, plus its top and bottom caps. Compute the flux of the vector field

$$\vec{F}(x, y, z) = \begin{pmatrix} -\sin \pi y \\ -\cos \pi x \\ xy \end{pmatrix}$$

both directly and by using the divergence theorem.

- Let $\vec{F}(x,y,z) = (x^2, yz, xz)$ and evaluate $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where S is the unit sphere...



4 Challenge

Let $\vec{F}(\vec{r}) = c\vec{r}/|\vec{r}|^3$ for some constant c and $\vec{r} = \langle x, y, z\rangle$. Show that the flux of $\vec{F}$ across a sphere S with center the origin is independent of the radius of S.

5 True/False

(a) T F Consider a sphere lying inside the smallest possible cylinder that can contain it. Then the ratio of the volume of the sphere to the volume of the cylinder is the same as the ratio of the surface areas.

(b) T F If we interchange the variables u, v in a parametrization of a surface (e.g. $\vec{r}(u,v) = \langle u^2, uv, v^2 - u\rangle$ becomes $\vec{r'}(u,v) = \langle v^2, uv, u^2 - v\rangle$) the surface this describes stays the same.

(c) T F If we interchange the variables u, v as above and use $\vec{r'}$ to compute a vector surface integral, we will get the same result as if using $\vec{r}$.

(d) T F Consider the **unit** normal vector of a parametrized surface (which can be obtained as $\vec{N}(u,v) = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$). The u- and v-derivatives of $\vec{N}$ are always parallel to the surface.

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.